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Evaluation of Correlated Bias Approximations in Experimental Uncertainty Analysis

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A new method to approximate the effect of correlated bias errors in experimental uncertainty analysis is presented. This new method is shown to be greatly superior to previously published and historical approximations, especially when bias errors are estimated in terms of a percentage of reading. To establish this method, the estimation of the bias limit for experimental results determined from measured variables containing biases from several elemental sources that are partially or wholly correlated is investigated using a Monte Carlo simulation. For each of four sample data reduction equations, the percentage coverage of the uncertainty limits computed using the new and previous approximate methods is determined for various combinations of elemental bias source correlation. These coverage values are compared with the desired coverage of 95% to see which method is the most consistent. The new method is found to be by far the most consistent method to approximate the effect of correlated bias errors.

Introduction

In nearly all experiments, the experimental result is not measured directly but is computed from measured values of different variables using a data reduction equation (DRE) or algorithm. A good example is the experimental determination of drag coefficient for a particular vehicle configuration in a wind-tunnel test. Defining drag coefficient in terms of the variables drag force F_D, density ρ, velocity V, and reference area A,

$$C_D = \frac{2F_D}{\rho V^2 A} \tag{1}$$

one can envision that errors in the values of the variables on the right-hand side of Eq. (1) will cause errors in the experimental result C_D.

A more general representation of a data reduction equation is

$$r = r(X_1, X_2, \ldots, X_J) \tag{2}$$

where r is the experimental result determined from J measured variables X_i. Each of the measured variables contains bias errors and precision errors. These errors in the measured values then propagate through the data reduction equation, thereby generating the bias and precision errors in the experimental result r.

In a recently issued AIAA Standard[1] some engineering approximations to the uncertainty propagation equations are presented that lead to a more easily usable large sample uncertainty analysis methodology. In this article, our interest lies in examining the effectiveness of several different approximations of the correlated bias uncertainty terms, and so from this point we will consider only the large sample equation for the 95% confidence bias limit in the result B_r,

$$B_r^2 = \sum_{i=1}^{J} \theta_i^2 B_i^2 + 2\sum_{i=1}^{J-1} \sum_{k=i+1}^{J} \theta_i \theta_k B_{ik} \tag{3}$$

using the notation that

$$\theta_i = \frac{\partial r}{\partial X_i} \tag{4}$$

The terms B_i represent the bias limits for each of the measured variables, and B_{ik} is the 95% confidence estimate of the covariance (or correlation) of the bias errors in variables X_i and X_k. This correlation term is often written as

$$B_{ik} = \rho_{bik} B_i B_k \tag{5}$$

where ρ_{bik} is the correlation coefficient appropriate for the bias errors in variables X_i and X_k. Equation (3) is an approximate equation. Its derivation and inherent assumptions are discussed in detail by Coleman and Steele.[2]

A bias limit B is defined as a 95% confidence estimator of β, the true but unknown bias error. It is assumed that corrections have been made for all bias errors whose values are known—thus, the remaining bias errors are equally as likely to be positive as negative. A 95% confidence estimate is interpreted as the experimenter being 95% confident that the true value of the bias error, if known, would fall within $\pm B$. As was discussed in Ref. 1, a useful approach to estimating the magnitude of a bias error is to assume that the bias error for a given case is a single realization drawn from some statistical parent distribution of possible bias errors. If bias error distributions are Gaussian with the standard deviation of the ith distribution being b_i, then

$$B_i = 2b_i \tag{6}$$

and

$$B_{ik} = \rho_{bik}(2b_i)(2b_k) \tag{7}$$

The bias limit is sometimes referred to as the systematic uncertainty.

In estimating the bias limit B_i for a variable X_i, it is usually helpful to separate the bias errors that influence the measurement of the variable into different elemental categories—calibration errors, data

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acquisition errors, data reduction errors, test technique errors, etc. Within each category, there may be several elemental sources of bias. For instance, if for the Jth variable, X_J, there are M elemental bias errors identified as significant and whose bias limits are estimated as $(B_J)_1, (B_J)_2, \ldots, (B_J)_M$, then the bias limit for X_J is calculated as the rss combination of the elemental limits

$$B_J = \left[\sum_{k=1}^{M}(B_J)_k^2\right]^{\frac{1}{2}} \tag{8}$$

It is not unusual for the uncertainties in the results of experimental programs to be influenced by the effects of correlated bias errors in the measurements of several of the variables. A typical example occurs when different variables are measured using the same transducer, such as multiple pressures sequentially ported to and measured with the same transducer or temperatures at different positions in a flow measured with a single probe that is traversed across the flowfield. Obviously, the bias errors in the variables measured with the same transducer are not independent of one another. Another common example occurs when different variables are measured using different transducers that have been calibrated against the same standard, a situation typical of electronically scanned pressure (ESP) measurement systems. In such a case, at least a part of the bias error arising from the calibration procedure will be the same for each transducer, and thus some of the elemental bias error contributions in the measurements of the variables will be correlated.

For example, consider

$$r = r(X_1, X_2, X_3) \tag{9}$$

where it is possible for portions of the bias limits B_1, B_2, and B_3 to arise from the same source(s). Application of Eq. (3) gives

$$B_r^2 = \theta_1^2 B_1^2 + \theta_2^2 B_2^2 + \theta_3^2 B_3^2 + 2\theta_1\theta_2\rho_{b12}B_1B_2 + 2\theta_1\theta_3\rho_{b13}B_1B_3 + 2\theta_2\theta_3\rho_{b23}B_2B_3 \tag{10}$$

If the measurements of X_1 and X_2 are each influenced by four elemental error sources and sources 2 and 3 are the same for both X_1 and X_2, and X_3 is influenced by three elemental error sources that are independent of the sources influencing X_1 and X_2 (so that $\rho_{b13} = \rho_{b23} = 0$), then

$$B_1^2 = (B_1)_1^2 + (B_1)_2^2 + (B_1)_3^2 + (B_1)_4^2 \tag{11}$$

$$B_2^2 = (B_2)_1^2 + (B_2)_2^2 + (B_2)_3^2 + (B_2)_4^2 \tag{12}$$

$$B_3^2 = (B_3)_1^2 + (B_3)_2^2 + (B_3)_3^2 \tag{13}$$

and the covariance (or correlated bias error) term containing

$$\rho_{b12}B_1B_2 = B_{12} \tag{14}$$

must be approximated. There is, in general, no way to obtain the data with which to make a statistical estimate of the correlation coefficient ρ_{b12}. It is the approximation of terms such as that in Eq. (14) that is the subject of this article.

Approximation of the Correlation Terms

Three approximations of the correlated bias error term were considered in this study. Historically, there have been two approximations used. The first ignored any contribution to the overall uncertainty due to correlated error sources. The uncertainty propagation equation for this approximation will be termed the no correlation term method. The second approximation[3] assumes perfect correlation of elemental error sources that are common to more than one variable, and the correlation term is approximated by the product of the rss combinations of the elemental errors for the variables with correlated error sources. This method will be presented in more detail later and will be termed the rss method. The third method of approximating the correlation term was developed after the initial results of the Monte Carlo simulation indicated the rss method sometimes failed to produce the desired coverage. This method uses an estimate of the covariance estimator B_{ik} rather than an estimate of the correlation coefficient ρ_{bik}. The uncertainty propagation equation for this approximation will be termed the sum of products (SOP) method.

Each of these uncertainty propagation equation approximations is described next using as an example the temperature difference experiment

$$\Delta T = T_2 - T_1 \tag{15}$$

in which the bias errors in each of the temperature measurements arise from four elemental bias error sources, with sources 2 and 3 being the same for T_2 and T_1.

No Correlation Term Approximation

If, as has usually been done in the past,[4,5] any effect of correlated bias uncertainties is ignored and the correlation coefficient is set equal to zero ($\rho_{bik} = 0$), the resulting equation is simply

$$B_r^2 = \sum_{i=1}^{J}\theta_i^2 B_i^2 \tag{16}$$

Application to Eq. (15) gives

$$B_{\Delta T}^2 = (1)^2 B_{T_2}^2 + (-1)^2 B_{T_1}^2 \tag{17}$$

where

$$B_{T_2}^2 = \left(B_{T_2}\right)_1^2 + \left(B_{T_2}\right)_2^2 + \left(B_{T_2}\right)_3^2 + \left(B_{T_2}\right)_4^2 \tag{18}$$

and

$$B_{T_1}^2 = \left(B_{T_1}\right)_1^2 + \left(B_{T_1}\right)_2^2 + \left(B_{T_1}\right)_3^2 \tag{19}$$

Root Sum Square Approximation

It is assumed, for measured variables X_i and X_k, that errors arising from the same elemental source are perfectly correlated. The approximation is then

$$\rho_{bik}B_iB_k = (1)B_i'B_k' \tag{20}$$

so that

$$B_r^2 = \sum_{i=1}^{J}\theta_i^2 B_i^2 + 2\sum_{i=1}^{J-1}\sum_{k=i+1}^{J}\theta_i\theta_k(1)B_i'B_k' \tag{21}$$

where B_i' and B_k' are the portions of B_i and B_k that arise from identical sources (and are therefore presumably perfectly correlated) and are given by

$$B_i' = \sqrt{\sum_{\alpha=1}^{L}(B_i)_\alpha^2} \tag{22}$$

where there are L common error sources for variables X_i and X_k. The other factor B_k' is given by an equation analogous to Eq. (22).

Application of Eqs. (21) and (22) to Eq. (15) gives

$$B_{\Delta T}^2 = (1)^2 B_{T_2}^2 + (-1)^2 B_{T_1}^2 + 2(1)(-1)B_{T_2}'B_{T_1}' \tag{23}$$

where

$$B_{T_2}' = \sqrt{\left(B_{T_2}\right)_2^2 + \left(B_{T_2}\right)_3^2} \tag{24}$$

and

$$B_{T_1}' = \sqrt{\left(B_{T_1}\right)_2^2 + \left(B_{T_1}\right)_3^2} \tag{25}$$

Sum of Products Approximation

Again it is assumed that error sources i and k are correlated, but the covariance estimator B_{ik} is approximated directly without consideration of the correlation coefficient. The bias limit equation becomes

$$B_r^2 = \sum_{i=1}^{J}\theta_i^2 B_i^2 + 2\sum_{i=1}^{J-1}\sum_{k=i+1}^{J}\theta_i\theta_k B_{ik} \tag{26}$$

where

$$B_{ik} = \sum_{\alpha=1}^{L} (B_i)_\alpha (B_k)_\alpha \tag{27}$$

is summed over the L elemental systematic error sources that are common for measurements of the variables X_i and X_k. The form of Eq. (27) was initially proposed based on the form of the term expressing the covariance of the bias errors in X_i and X_k as derived in Ref. 2. As noted by a reviewer, the form can also be derived as shown in the Appendix.

Application to Eq. (15) gives

$$B_{\Delta T}^2 = (1)^2 B_{T_2}^2 + (-1)^2 B_{T_1}^2 + 2(1)(-1) B_{T_2 T_1} \tag{28}$$

where

$$B_{T_1 T_2} = \left(B_{T_1}\right)_2 \left(B_{T_2}\right)_2 + \left(B_{T_1}\right)_3 \left(B_{T_2}\right)_3 \tag{29}$$

When all elemental error sources are a fixed value (percent of full scale), it can be shown that this method is algebraically equivalent to the rss method; however, that is not the case when some of the common elemental sources are of the percent of reading type or vary with the variable value in some other manner.

Monte Carlo Simulation

Figure 1 shows the logic scheme used to conduct the Monte Carlo simulation, which was patterned after a technique used previously.[6] "True" values for each variable in the data reduction equation were selected, and the true value for the result was calculated. (The word true is emphasized to indicate that it represents the actual physical quantity of the parameter if it could be measured without any bias error or precision error, which is always unobtainable.) The 2σ bias limits for each elemental error source were then assigned assuming that the individual error sources were normally distributed, and a random value for each error source was found using a Gaussian random deviate generator subroutine with the assigned 2σ bias limit for each error source. (Note that, strictly, 2σ bias limits correspond to about a 95.5% confidence level.) The Gaussian deviates had a mean of zero and obviously an equal probability of being positive or negative. When the elemental errors for the variables were correlated, the same elemental error value was used for each variable. The individual random elemental error values were then summed and added to the true value for each variable, and these variable values were then used in the data reduction equation to obtain the random value of the result. This value represents the result of the experiment when the bias errors are present (the value that would be calculated from the experimental data if all precision errors were zero).

A 95% confidence bias limit for the result was calculated from each uncertainty analysis approximation equation. An interval of $\pm B_r$ was placed around the random result, and if the true result was found to be within the interval, a counter was incremented. This procedure was repeated 10,000 times, and the percent coverage, or fraction of time the true result was within the estimated interval, was determined for each approximation. A bias limit ratio, defined as B_r(calc)/B_r(actual), was also computed for each approximation. The term B_r(calc) is the average of the 10,000 B_r calculated for a given approximation, whereas B_r(actual) is twice the standard deviation of the distribution of the 10,000 results. (In preliminary investigations, comparison of results of simulations using 10^3–10^5 iterations led to the choice of 10^4 as suitable.)

The Monte Carlo simulation can be interpreted as representing what would happen if a "different but identical" experimental apparatus were set up at 10,000 different locations, with each of the elemental error sources estimated at a 95% confidence value, but with the actual value of each particular elemental error source different from one location to the next. The three uncertainty analysis approximation equations would each provide an estimate of the bias limit of the result for each experimental apparatus. Since each of the elemental error sources is specified at 95% confidence, the uncertainty in the result is also desired to have 95% confidence.

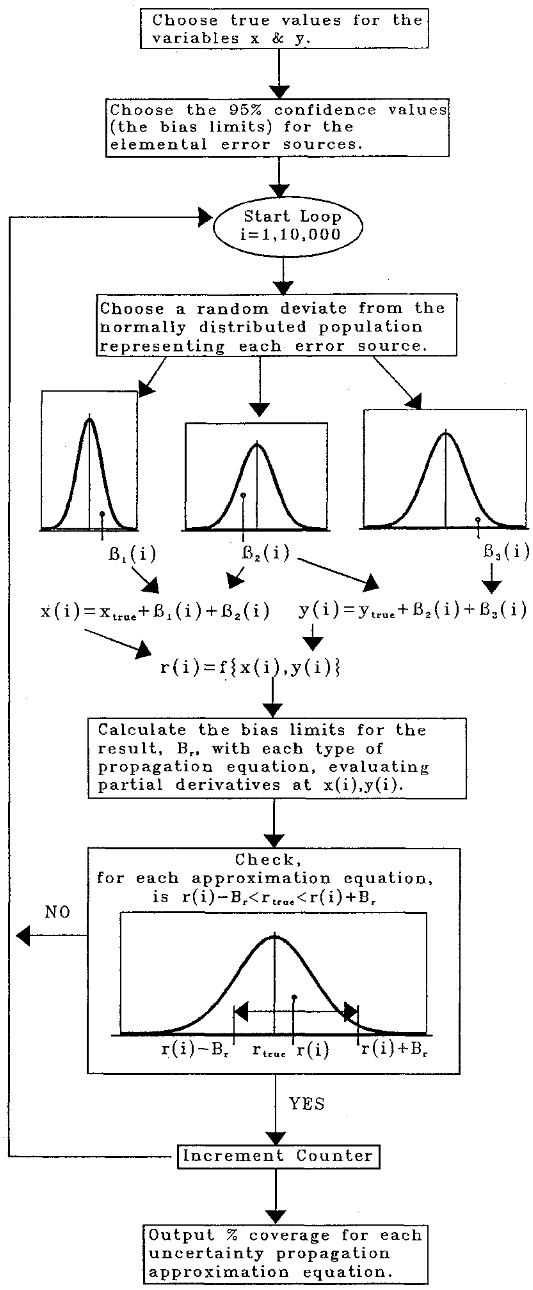

Fig. 1 Monte Carlo simulation flow chart.

Results and Discussion

To study the effectiveness of each of the three correlation term approximations, the Monte Carlo simulation was applied to four hypothetical experiments with data reduction equations (DREs) that provide both linear and nonlinear cases. These DREs were chosen to represent a range of typical engineering experiments, and realistic ranges for the values of the bias limits of the elemental error sources were used. The elemental error sources were allowed to be both percent of full scale and percent of reading type errors. The examples that are presented include data reduction equations for the difference in two temperatures, the average of three temperatures, pressure coefficient, and compressor efficiency. Each of these is presented, in turn, with a description of the hypothetical experiment.

Example 1

The first data reduction equation to be studied is the difference in two temperatures determined from two thermocouple measurements

$$\Delta T = T_2 - T_1 \tag{30}$$

Two dominant error sources are assumed for each thermocouple—the first from the data acquisition system and the second from the

Input Data		
DRE: $r=\Delta T=T_2-T_1$ T_2=350 K , T_1=300 K ; ΔT_{true}=50 K		
	B_{T2} error source 1 error source 2	B_{T1} error source 1 error source 2
Experiment 1	3.5 K 0.1%(0.35 K)	3.5K 0.1%(0.30K)
Experiment 2	3.5 K 1.0% (3.5 K)	3.5 K 1.0% (3.0 K)
Experiment 3	0.35 K 1.0% (3.5 K)	0.35 K 1.0% (3.0 K)

Results			
% Coverage	Type of Approximation		
	No Corr	RSS	SOP
Experiment 1	100.0	15.4	95.7
Experiment 2	100.0	82.9	95.7
Experiment 3	100.0	95.5	95.7
Bias Limit Ratio $\frac{B_{\Delta T}(\text{calc})}{B_{\Delta T}(\text{actual})}$	Type of Approximation		
	No Corr	RSS	SOP
Experiment 1 $B_{\Delta T}$(act)=0.0498K	99.9	0.094	1.00
Experiment 2 $B_{\Delta T}$(act)=0.498K	13.7	0.70	1.02
Experiment 3 $B_{\Delta T}$(act)=0.498K	9.26	0.994	1.00

Fig. 2 Input data and Monte Carlo results for example 1.

Input Data			
DRE: $r=T_{avg}=(T_1+T_2+T_3)/3$ T_1=343 K , T_2=347 K, T_3=352 K T_{avg}(true)=347.3 K			
	B_{T1} source 1 source 2	B_{T2} source 1 source 2	B_{T3} source 1 source 2
Experiment 1	3.5 K 0.1% (0.34 K)	3.5K 0.1% (0.35K)	3.5K 0.1% (0.35K)
Experiment 2	3.5 K 1.0% (3.43K)	3.5 K 1.0% (3.47K)	3.5 K 1.0% (3.52K)
Experiment 3	0.35 K 1.0% (3.43K)	0.35 K 1.0% (3.47K)	0.35 K 1.0% (3.52K)

Results			
% Coverage	Type of Approximation		
	No Corr	RSS	SOP
Experiment 1	75.9	95.8	95.8
Experiment 2	81.7	98.0	98.0
Experiment 3	88.3	99.4	99.4
Bias Limit Ratio $\frac{B_{T_{AVG}}(\text{calc})}{B_{T_{AVG}}(\text{actual})}$	Type of Approximation		
	No Corr	RSS*	SOP*
Experiment 1 $B_{T_{avg}}$(act)=3.452 K	0.59	1.02	1.02
Experiment 2 $B_{T_{avg}}$(act)=4.257 K	0.68	1.16	1.16
Experiment 3 $B_{T_{avg}}$(act)=2.580 K	0.79	1.35	1.35

* The bias limits calculated with the RSS and SOP approximations differ past the 3 rd significant digit.

Fig. 3 Input data and Monte Carlo results for example 2.

reference thermometer used in calibration. (Note that other sources, such as installation errors not accounted for in the calibration procedure, are often present in real experiments.) The data acquisition error is specified as a fixed value and the calibration error as a percentage of reading. If it is postulated that the thermocouples share the same acquisition system and are calibrated with the same thermometer, then it can be reasonably argued that the errors are completely correlated. Three separate experiments were studied—one in which the acquisition system error was an order of magnitude greater than the calibration error, another in which the errors were of the same magnitude, and a third where the calibration error was an order of magnitude greater than the acquisition system error. The hypothetical true values for the temperatures and the 95% confidence bias limits of the elemental error source values were assumed for each case. The input data used and the results from the Monte Carlo simulation are shown in Fig. 2.

The results for this DRE illustrate several key points. The sum of products approximation consistently provides the desired coverage and gives B_r estimates that are close to the correct value. The rss approximation does not consistently provide the desired coverage and, for the particular elemental error values used in experiment 1, gave a B_r estimate that was one-tenth the correct value. The approximation that ignores the correlated errors consistently provides 100% coverage, but with B_r estimates that are up to 100 times too large!

Example 2

The second data reduction equation studied was the average of three temperature measurements

$$T_{\text{avg}} = \frac{T_1 + T_2 + T_3}{3} \tag{31}$$

Each temperature measurement was assumed to have the same two elemental error sources as in example 1, and the same three experiments were examined. It was assumed that there was complete correlation in the bias errors for all three measurements, thus correlation terms for T_1 and T_2, T_1 and T_3, and T_2 and T_3 were included.

The input data and results for this DRE are shown in Fig. 3, respectively. The approximation that neglects correlated error sources

Input Data

DRE: $r = C_p = 2\frac{RT}{P_1V^2}(P_2 - P_1)$

T_1=300 K P_1= 101.3 kPa V=174m/s
P_2=80.5 kPa
C_p(true)=-1.167

	B_{T2} source 1 source 2	B_{P1} source 1 source 2	B_{P2} source 1 source 2	B_V source 1
Exp. 1	0.0 K 0.0%	1.0kPa 1.0% 1.01kPa	1.0kPa 1.0% 0.80kPa	0.0 m/s
Exp. 2	0.5 K 0.25% (0.75 K)	1.0kPa 1.0% 1.01kPa	1.0kPa 1.0% 0.80kPa	0.5 m/s
Exp. 3	0.5 K 0.25% (0.75K)	5.0 1.0% 1.01kPa	5.0kPa 1.0% 0.80kPa	2.0 m/s

Results

% Coverage	Type of Approximation		
	No Corr	RSS	SOP
Experiment 1	100.0	19.2	95.4
Experiment 2	100.0	90.2	95.3
Experiment 3	100.0	94.9	95.2

Bias Limit Ratio $\frac{B_{C_p}\text{(calc)}}{B_{C_p}\text{(actual)}}$	Type of Approximation		
	No Corr	RSS	SOP
Experiment 1 B_{C_p}(act)=0.00115	56.6	0.122	1.01
Experiment 2 B_{C_p}(act)=0.01378	7.08	0.83	1.00
Experiment 3 B_{C_p}(act)=0.06372	5.76	0.99	1.00

Fig. 4 Input data and Monte Carlo results for example 3.

again fails to provide the desired coverage and in this case underestimates B_r. Both the rss and the sum of products approximations provide the same bias limits and percent coverage out to the third significant figure. Closer examination showed that when the data reduction equation is linear and additive, the rss and the SOP methods become mathematically very similar and thus provide almost identical bias limits. In the experiments studied, both the rss and the SOP method provided at least the required 95% coverage. In the cases where the coverage was greater than 95%, the overestimates of B_r were not significant from an engineering viewpoint, particularly when compared with the orders of magnitude overestimates in example 1 for the no correlation term approximation.

Example 3

The third data reduction equation studied was the calculation of the pressure coefficient for a model in a wind tunnel,

$$C_p = \left(2RT_1/P_1V_1^2\right)(P_2 - P_1) \tag{32}$$

Input Data

DRE: $r = \eta = \dfrac{\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1}{\frac{T_2}{T_1} - 1}$

T_1= 294 K, P_1=101.3 kPa, γ=1.4
T_2= 533 K, P_2=658.2 kPa
η(true) = 0.8696

	B_{T1} (1) (2)	B_{T2} (1) (2)	B_{P1} (1) (2)	B_{P2} (1) (2)	B_γ (1)
Exp. 1	0.5K 2.5% (7.35K)	0.5K 2.5% (13.3)	0.5kPa 2.5% (2.53)	0.5kPa 2.5% (16.5)	0.0
Exp. 2	0.5K 2.5% (7.35)	0.5K 2.5% (13.3)	0.5kPa 2.5% (2.53)	0.5 2.5% (16.5)	.01
Exp. 3	3.0K 1.0% (2.94)	3.0K 1.0% (5.33)	1.0kPa 1.0% (1.01)	1.0kPa 1.0% (6.58)	0.0

Results

% Coverage	Type of Approximation		
	No Corr	RSS	SOP
Experiment 1	100.0	17.0	95.7
Experiment 2	100.0	95.3	95.7
Experiment 3	100.0	76.4	95.7

Bias Limit Ratio $\frac{B_\eta\text{(calc)}}{B_\eta\text{(actual)}}$	Type of Approximation		
	No Corr	RSS	SOP
Experiment 1 B_η(act)=0.0029	25.5	0.124	1.00
Experiment 2 B_η(act)=0.0201	3.80	1.00	1.01
Experiment 3 B_η(act)=0.0101	3.71	0.59	1.01

Fig. 5 Input data and Monte Carlo results for example 4.

Specifying that the instrumentation consists of a thermocouple, pressure transducers, and a hot-wire anemometer, the temperature, pressure, and velocity upstream and the pressure at a point on the surface of the body can be measured. The pressure coefficient at that point on the body can then be determined. Each measurement has a systematic uncertainty (bias) made up of various elemental sources, and it is assumed the two pressure transducers share the same elemental error sources. For example, elemental error source 1 could be a fixed value from the data acquisition system and error source 2 could be the uncertainty remaining after calibration and be some percentage of reading. The value of the gas constant R is assumed known with negligible uncertainty.

Figure 4 shows the input data and Monte Carlo results for this example. The results are very similar to those seen in example 1, and similar conclusions can be drawn. Ignoring the effects of the correlated errors again leads to B_r estimates that are enormously in error, the rss approximation results are not consistent, and the SOP approximation gives very satisfactory results.

Example 4

The fourth data reduction equation studied was the determination of efficiency for an air compressor

$$\eta = \frac{(P_2/P_1)^{[(\gamma-1)/\gamma]} - 1}{(T_2/T_1) - 1} \tag{33}$$

The compressor inlet pressure and temperature (P_1 and T_1) and outlet pressure and temperature (P_2 and T_2) must be measured, and a value for the specific heat ratio γ must be specified. It is assumed that the bias errors for the pressure transducers are correlated and that the bias errors for the thermistors are correlated, but there is no correlation between the temperature measurement errors and the pressure measurement errors.

The input data and results from the Monte Carlo simulation are shown in Fig. 5, and the conclusions that can be drawn are identical to those for examples 1 and 3.

Conclusion

Monte Carlo simulations for the four data reduction equations investigated showed that the sum of products approximation for correlated bias error effects produced B_r estimates that provided the desired coverage much more consistently than the no correlation term and the rss approximations. This behavior seems to hold for any given level of confidence chosen—a check using 99% confidence estimates for the elemental error sources supported the same conclusions drawn using the 95% confidence estimates. This sum of products method, newly proposed in this article, is recommended as the approximation that should be used when one must account for correlated bias error effects.

The method that neglects any correlated error effects usually provides 100% coverage, but this is not as desirable as it sounds, since it involves large overestimates of the bias limits. In addition, for certain types of data reduction equations, neglecting correlation effects underestimates the bias limit and provides less than the desired 95% coverage.

Appendix

Using nomenclature similar to that shown in Fig. 1, let the bias errors β be random variables and the measured values of x and y be

$$x = x_{\text{true}} + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 \tag{A1}$$

$$y = y_{\text{true}} + c_2\beta_2 + c_3\beta_3 + c_4\beta_4 \tag{A2}$$

where the various a and c are scaling constants and where x and y are both obviously affected by elemental sources 2 and 3. The covariance of the measured values of x and y is

$$\sigma_{xy} = E\{[x - E(x)][y - E(y)]\} \tag{A3}$$

Substituting Eqs. (A1) and (A2) into Eq. (A3), one obtains

$$\sigma_{xy} = a_2c_2E[\beta_2 - E(\beta_2)]^2 + a_3c_3E[\beta_3 - E(\beta_3)]^2 \tag{A4}$$

or

$$\sigma_{xy} = a_2c_2\sigma^2_{\beta_2} + a_3c_3\sigma^2_{\beta_3} \tag{A5}$$

Rewriting this as

$$\sigma_{xy} = \left(a_2\sigma_{\beta_2}\right)\left(c_2\sigma_{\beta_2}\right) + \left(a_3\sigma_{\beta_3}\right)\left(c_3\sigma_{\beta_3}\right) \tag{A6}$$

produces a form analogous to that of Eq. (27).

References

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